

Aspects of the Bosonic Spectral Action

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Abstract. A brief description of the elements of noncommutative spectral geometry as an approach to unification is presented. The physical implications of the doubling of the algebra are discussed. Some high energy phenomenological as well as various cosmological consequences are presented. A constraint in one of the three free parameters, namely the one related to the coupling constants at unification, is obtained, and the possible rôle of scalar fields is highlighted. A novel spectral action approach based upon zeta function regularisation, in order to address some of the issues of the traditional bosonic spectral action based on a cutoff function and a cutoff scale, is discussed.

1. Introduction

The Standard Model (SM) of strong and electroweak interactions remains the most successful particle physics model we have at hand and its validity has been recently confirmed by the discovery of the Higgs boson. However, several conceptual questions remain unanswered, while it may be necessary to go beyond the SM, possibly relating it to a theory of Quantum Gravity. Noncommutative Spectral Geometry (NCSG) aims at explaining some of the conceptual issues of the SM, whilst it offers a new geometrical framework to address physics at the Quantum Gravity regime. To construct a Quantum theory of Gravity coupled to matter, one may either neglect matter altogether (as for instance within the framework of Loop Quantum Gravity), or consider instead that the interaction between gravity and matter is the most important ingredient to define the dynamics. Noncommutative Spectral Geometry [1, 2] follows the latter approach, aiming at defining the noncommutative algebra of observables of a Quantum theory of Gravity.

Noncommutative spectral geometry starts from the following remark. At energy scales much below the Planck scale it is reasonable to assume that physics can be described by continuum fields and an effective action (the sum of the Einstein-Hilbert and SM actions), but close to Planck scale this assumption is no longer valid and Quantum Gravity effects may imply that space-time is a heavily noncommutative manifold. Remaining close but below Planck scale, one can however consider that the algebra of coordinates is only a lightly noncommutative algebra of matrix valued functions, and by chosen properly this algebra NCSG leads to a purely geometric explanation of the SM coupled to gravity [3]. In the context of NCSG, gravity and the SM fields are packaged into geometry and matter on a Kaluza-Klein noncommutative space and using well-established experimental results at the electroweak scale, we can guess the small-scale space-time structure avoiding an *ad hoc* proposal. In that sense, noncommutative geometry can be considered as a bottom-up approach, complimentary to the top-down string theory approach.

Noncommutative spectral geometry proposes to consider the SM as a phenomenological model which dictates the space-time geometry. In this way, the geometric space is defined as the product of a 4-dimensional compact Riemannian manifold \mathcal{M} , describing the geometry of space-time, with an internal zero-dimensional discrete finite internal Kaluza-Klein space \mathcal{F} , describing the internal geometry, attached to each point. Such simple noncommutative spaces $\mathcal{M} \times \mathcal{F}$, where the noncommutative algebra describing space is the algebra of functions over ordinary space-time, are called *almost commutative manifolds*. Note that such spaces are different from the more general noncommutative spaces such as the Moyal plane for which $[x^i, x^j] = i\theta^{ij}$, where θ^{ij} is an anti-symmetric real $d \times d$ matrix (with d the space-time dimensionality) representing the fuzziness of space-time. In NCSG, the description of ordinary Riemannian manifolds in terms of spectral data, is extended for noncommutative manifolds. Hence, one defines the almost commutative manifold $\mathcal{M} \times \mathcal{F}$ by a spectral triple, and dynamics are given by a spectral action that sums up all frequencies of vibration of space. At last, one aims at answering whether we can hear the shape of such a spectral triple, called a *spinorial drum*.

In the following, we will give a short description of the main elements of NCSG [4]-[7], discuss its phenomenological particle physics consequences, the physical implications of the precise construction of the almost commutative manifold [8, 9], and examine the cosmological consequences of the gravitational sector of the theory [10]-[18]. We will then address some issues regarding the traditional bosonic spectral action approach and highlight a new proposal [19].

2. Elements of Noncommutative Spectral Geometry

Let us start with the case of spin (in order to be able to describe spinors) manifolds in NCSG. Given a compact 4-dimensional Riemannian spin manifold \mathcal{M} , consider the set $C^\infty(\mathcal{M})$ of smooth infinitely differentiable functions, and the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S)$ of square-integrable spinors S on \mathcal{M} . One can show that the set $C^\infty(\mathcal{M})$ is an algebra \mathcal{A} under point-wise multiplication, acting on \mathcal{H} as multiplication operators. Let us also consider \mathcal{D} , the Dirac operator $-i\gamma^\mu \nabla_\mu^s$, acting as first order differential operator on the spinors. The algebra, Hilbert space and Dirac operator form the canonical triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \mathcal{D})$. In addition, we consider the γ_5 operator with $\gamma_5^2 = 1, \gamma_5^* = \gamma_5$, which is just a \mathbb{Z}_2 -grading, that decomposes the Hilbert space \mathcal{H} into a positive and negative eigenspace $L^2(\mathcal{M}, S) = L^2(\mathcal{M}, S)^+ \oplus L^2(\mathcal{M}, S)^-$, hence playing the rôle of a chirality operator. We also consider an antilinear isomorphism $J_{\mathcal{M}}$ with $J_{\mathcal{M}}^2 = -1, J_{\mathcal{M}}\mathcal{D} = \mathcal{D}J_{\mathcal{M}}, J_{\mathcal{M}}\gamma_5 = \gamma_5J_{\mathcal{M}}$, as the charge conjugation operator on spinors.

Consider now an almost-commutative manifold $\mathcal{M} \times \mathcal{F}$. The canonical triple defining \mathcal{M} encodes the space-time structure, whereas the triple $(\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})$ encodes the internal degrees of freedom at each point in space-time, allowing a description of a gauge theory on the spin manifold \mathcal{M} . To obtain the SM, the most important ingredient is the choice of the matrix algebra $\mathcal{A}_{\mathcal{F}}$, acting on the Hilbert space $\mathcal{H}_{\mathcal{F}}$ via matrix multiplication. The operator $\mathcal{D}_{\mathcal{F}}$ is a 96×96 matrix expressed in terms of the 3×3 Yukawa mixing matrices and a real constant responsible for the neutrino mass terms. This operator corresponds to the inverse of the Euclidean propagator of fermions. In addition, we consider a $\gamma_{\mathcal{F}}$ grading such that $\gamma_{\mathcal{F}} = +1$ for left-handed fermions and $\gamma_{\mathcal{F}} = -1$ for right-handed ones, and a conjugation operator $J_{\mathcal{F}}$ for the finite space \mathcal{F} . The almost-commutative manifold $\mathcal{M} \times \mathcal{F}$ is expressed by the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$:

$$\mathcal{M} \times \mathcal{F} := (C^\infty(\mathcal{M}, \mathcal{A}_{\mathcal{F}}), L^2(\mathcal{M}, S) \otimes \mathcal{H}_{\mathcal{F}}, \mathcal{D} \otimes \mathbb{I} + \gamma_5 \otimes \mathcal{D}_{\mathcal{F}}) .$$

The choice of the algebra $\mathcal{A}_{\mathcal{F}}$ is the most important input of the NCSG approach to the SM, and has to be chosen appropriately. For instance, $\mathcal{A}_{\mathcal{F}}$ cannot be right-handed symmetric. It has been shown [20] that this algebra has to be the product of the algebra of quaternions and the algebra of the complex $k \times k$ matrices with k an even number $k = 2a$:

$$\mathcal{A}_{\mathcal{F}} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) .$$

The first value of k that produces the correct number of fermions (namely 16) in each of the three generations, is $k = 4$. Hence, NCSG predicts that the number of fermions is the square of an even integer, while the existence of three generations is just a physical input. Note that the particular choice of Hilbert space is of no importance, since all separable infinite-dimensional Hilbert spaces are isomorphic. Hence, the fermions of the SM provide the Hilbert space of a spectral triple for the algebra, while the boson of the SM, including the Higgs boson, are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry. Thus, the Higgs boson becomes just a gauge field corresponding to a finite difference.

To derive a physical Lagrangian one then applies the spectral action principle, stating that the action functional depends only on the spectrum of the fluctuated Dirac operator \mathcal{D}_A :

$$\mathcal{D}_A = \mathcal{D} + A + \epsilon' J A J^{-1} , \quad (1)$$

with $A = A^*$ a self-adjoint operator of the form

$$A = \sum_j a_j [\mathcal{D}, b_j] ; \quad a_j, b_j \in \mathcal{A} , \quad (2)$$

J an anti-unitary operator such that $J^2 = 1$ and $\epsilon \in \{\pm 1\}$, and is of the form

$$\text{Tr}(f(\mathcal{D}_A^2/\Lambda^2)) , \quad (3)$$

with f a cut-off function and Λ denoting the energy scale at which this Lagrangian is valid. More precisely, f is a positive function that falls to zero at large values of its argument, so that the integrals $\int_0^\infty f(u)u du$ and $\int_0^\infty f(u)du$ are finite. Typical cut-off functions f used in the literature are $f(x) = 1$ for $x \leq \Lambda$, or $f(x) = e^{-x}$. The action given in Eq. (3) above, sums up all eigenvalues of the fluctuated Dirac operator \mathcal{D}_A which are smaller than the cut-off energy scale Λ . This trace can be then evaluated using heat kernel techniques and thus expressed through the Seeley-de Witt coefficients a_n , known for any second order elliptic differential operator, as $\sum_{n=0}^\infty F_{4-n} \Lambda^{4-n} a_n$ where F is defined as $f(\mathcal{D}_A^2)$.

The spectral action can be expanded in powers of the scale Λ in the form [21]

$$\text{Tr} \left(f \left(\frac{\mathcal{D}_A^2}{\Lambda^2} \right) \right) \sim \sum_{k \in \text{DimSp}} f_{2k} \Lambda^{2k} \oint |\mathcal{D}_A|^{-2k} + f(0) \zeta_{\mathcal{D}_A^2}(0) + \mathcal{O}(1) , \quad (4)$$

where f_{2k} are the momenta of the function f , defined as

$$f_{2k} \equiv \int_0^\infty f(u) u^{2k-1} du , \quad \text{for } k > 0 ,$$

and $f_0 \equiv f(0)$. The noncommutative integration is defined in terms of residues of zeta functions, $\zeta_{\mathcal{D}_A^2}(s) = \text{Tr}(|\mathcal{D}_A|^{-2s})$ at poles of the zeta function, and the sum is over points in the *dimension spectrum* of the spectral triple.

Since f is a cut-off function, its Taylor expansion vanishes at zero, implying that the asymptotic expansion of the trace, namely

$$\text{Tr}(f(\mathcal{D}_A^2/\Lambda^2)) \sim 2f_4 \Lambda^4 a_0(\mathcal{D}_A^2) + 2f_2 \Lambda^2 a_2(\mathcal{D}_A^2) + f(0) a_4(\mathcal{D}_A^2) + \mathcal{O}(\Lambda^{-2}) , \quad (5)$$

can be only given from the three first terms of the expansion. The cut-off function plays a rôle through only three of its momenta:

$$f_4 = \int_0^\infty f(u) u^3 du ; \quad f_2 = \int_0^\infty f(u) u du ; \quad f_0 = f(0) . \quad (6)$$

related to the cosmological constant, the gravitational constant and the coupling constants at unification, respectively.

The bosonic spectral action, Eq. (3), must be seen *à la* Wilson, hence as the bare action at the mass scale Λ . This action only accounts for the bosonic part. Hence, to account for the terms involving fermions and their coupling to the bosons, one needs to include the fermionic part, which for a KO-dimension 2 almost commutative manifold reads

$$(1/2)\langle J\Psi, \mathcal{D}_A\Psi \rangle ; \Psi \in \mathcal{H}^+ . \quad (7)$$

After a long calculation one eventually obtains that the bosonic spectral action at the cutoff scale Λ and using the cutoff normalisation through the cutoff function f , reads

$$S_\Lambda = \frac{-2af_2\Lambda^2 + ef_0}{\pi^2} \int |\phi|^2 \sqrt{g} d^4x + \frac{f_0}{2\pi^2} \int a |D_\mu \phi|^2 \sqrt{g} d^4x - \frac{f_0}{12\pi^2} \int a R |\phi|^2 \sqrt{g} d^4x \\ - \frac{f_0}{2\pi^2} \int \left(g_3^2 G_\mu^i G^{\mu i} + g_2^2 F_\mu^a F^{\mu a} + \frac{5}{3} g_1^2 B_\mu B^\mu \right) \sqrt{g} d^4x + \frac{f_0}{2\pi^2} \int b |\phi|^4 \sqrt{g} d^4x + \mathcal{O}(\Lambda^{-2}) , \quad (8)$$

with a, b, c, d, e constants depending on the Yukawa parameters. Adding to the above action the fermionic part, as indicated in Eq. (7), one obtains [3] the full SM Lagrangian. Since this spectral action is characterised by the cutoff function f and the cutoff scale Λ , we will call it the *cutoff bosonic spectral action*, to differentiate it from another regularisation procedure we will highlight later.

To discuss the particle physics phenomenological consequences of NCSG let us briefly discuss the obtained Lagrangian. Its coefficients are given in terms of the three momenta $f(0), f_2, f_4$ of the cut-off function f , of the cut-off scale Λ , of the vacuum expectation value of the Higgs field ϕ , and of the coefficients a, b, c, d, e , which are determined by the mass matrices in the Dirac operator \mathcal{D}_F . Given that among the various relations connecting the coefficients a, b, c, d, e , one finds $g_2^2 = g_3^2 = (5/3)g_1^2$, which holds in several Grand Unified Theories (GUTs) (like $SU(5)$), one may assume that the theory is valid at the GUT scale. One then uses standard renormalisation group flow techniques to obtain predictions for the SM phenomenology. For instance, one finds that the top quark mass is $m_t \leq 180$ GeV. The NCSG approach leads to a Higgs doublet with a negative mass term and a positive quartic term, hence implying the existence of a spontaneously symmetry breaking mechanism of the electroweak symmetry. Let us comment on the predicted value of the Higgs mass. The NCSG model involves three scalars, namely a Higgs field, a singlet and a dilaton. The singlet is a real scalar field associated with the Majorana mass of the right-handed neutrino, having a nontrivially mixing with the Higgs field. In the original approach [3], the singlet was integrated out being replaced by its vacuum expectation value, leading to an incorrect prediction of the Higgs mass, namely $167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}$. This conflict was resolved in the subsequent approach [22] where this assumption was relaxed. Hence considering the mixing between the Higgs doublet and singlet, consistency with the experimental result of a 125 GeV Higgs mass and a 170 GeV top quark mass was achieved. Note that the rôle of the singlet field was already mentioned previously [23]. Moreover, the experimentally found Higgs mass can be accommodated by either considering a model based on a larger symmetry, the *grand symmetry*, where the algebra is $A_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ [24], or by generalising the inner fluctuations to real spectral triples that fail on the first order condition, leading to a Pati-Salam type of model $SU(2)_R \times SU(2)_L \times SU(4)$ [25].

Assuming the big desert hypothesis, one-loop renormalisation group analysis for the three gauge couplings and the Newton constant, has shown [3] that they do not exactly meet at a point; the error being just a few percent. Hence, the big desert hypothesis is only approximately valid, and one may expect new physics between unification and present energy scales. Finally,

NCSG predicts the existence of a see-saw mechanism for neutrino masses with large right-handed neutrino mass of the order of the cutoff scale Λ .

In conclusion, NCSG offers an elegant geometric interpretation of the SM coupled to gravity. Applying Einstein's theory of General Relativity (GR) within Riemannian geometry, one obtains the familiar gravitational theory. As we have discussed above, applying the spectral action approach within the context of an almost commutative geometry, one gets gravity combined with Yang-Mills and Higgs. What remains to be done in this programme, is to construct the appropriate tools we need to apply within a fully noncommutative geometry and then deduce the theory to which they will lead us.

3. Physical meaning of the doubling of the algebra

Let us highlight the physical implications of choosing an almost commutative manifold. The geometry is specified by the product $\mathcal{M} \times \mathcal{F}$ given from the spectral triple

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma) = (C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D}_{\mathcal{M}}, J_{\mathcal{M}}, \gamma_5) \otimes (\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}, J_{\mathcal{F}}, \gamma_{\mathcal{F}}) ,$$

defined as

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma) = (\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1, J_1, \gamma_1) \otimes (\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2, J_2, \gamma_2) ,$$

with

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 , \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 , \quad \mathcal{D} = \mathcal{D}_1 \otimes 1 + \gamma_1 \otimes \mathcal{D}_2 , \quad \gamma = \gamma_1 \otimes \gamma_2 , \quad J = J_1 \otimes J_2 ,$$

where $J^2 = -1$, $[J, \mathcal{D}] = 0$, $[J_1, \gamma_1] = 0$ and $\{J, \gamma\} = 0$. The doubling of the algebra is intimately related to dissipation, gauge field structure (necessary to address the physics of the SM), as well as neutrino mixing, while it incorporates the seeds of quantisation [8, 9].

Consider the classical Brownian motion of a particle of mass m with equation of motion

$$m\ddot{x}(t) + \gamma\dot{x}(t) = f(t) , \tag{9}$$

where $f(t)$ denotes a random Gaussian distributed force. This equation of motion can be derived from a Lagrangian in a canonical procedure, using a delta functional classical constraint representation as a functional integral. It is easy to see [8] that the constraint condition at the classical level introduces a new coordinate, called y , with the y -system being the time-reversed of the x -one, so that the equations of motion read

$$m\ddot{x} + \gamma\dot{x} = f , \quad m\ddot{y} - \gamma\dot{y} = 0 . \tag{10}$$

The x -system represents an *open* (dissipating) system, while the $\{x, y\}$ -system is a *closed* one. This doubling, discussed here in a completely classical context, is necessary in order to build a canonical formalism for dissipative systems [8].

To argue the relation between the doubling of the algebra and the gauge field structure, consider

$$m\ddot{x} + \gamma\dot{x} + kx = 0 , \tag{11}$$

the equation of a classical one-dimensional damped harmonic oscillator, with time independent quantities m, γ, k . Following the previous discussion, we will complement the x -system with its time-reversed image, called y -system, as

$$m\ddot{y} - \gamma\dot{y} + ky = 0 , \tag{12}$$

in order to build a well-defined Lagrangian formalism. Equation (12) above is that of a one-dimensional amplified harmonic oscillator.

The Lagrangian of the closed $\{x, y\}$ -system can be then written as

$$L = \frac{1}{2m}(\dot{m}x_1 + \frac{e_1}{c}A_1)^2 - \frac{1}{2m}(\dot{m}x_2 + \frac{e_2}{c}A_2)^2 - \frac{e^2}{2mc^2}(A_1^2 + A_2^2) - e\Phi , \quad (13)$$

where we have introduced the coordinates x_1, x_2 through

$$x_1(t) = \frac{x(t) + y(t)}{\sqrt{2}} , \quad x_2(t) = \frac{x(t) - y(t)}{\sqrt{2}} , \quad (14)$$

and the vector potential

$$A_i = \frac{B}{2}\epsilon_{ij}x_j \quad \text{for } i, j = 1, 2 \quad \text{with } B \equiv \frac{\gamma c}{e} , \quad \epsilon_{ii} = 0 , \quad \epsilon_{12} = -\epsilon_{21} = 1 . \quad (15)$$

The Lagrangian (14) describes two particles having opposite charges $e_1 = -e_2 \equiv e$ in the potential $\Phi = \Phi_1 - \Phi_2$, where $\Phi_i \equiv (k/2/e)x_i^2$ in the constant magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Identifying the doubled coordinate with the x_2 , we observe that it acts as the gauge field component A_1 to which the original x_1 coordinate is coupled. We hence conclude that energy dissipated by one of the two systems is gained by the other one, so that the gauge field can be seen as the *reservoir* in which the system is embedded.

Dissipation may also lead to a quantum evolution. This can be easily shown by using 't Hooft's conjecture, saying that loss of information (i.e., dissipation) in a regime of deterministic dynamics may lead to a quantum mechanical evolution. We consider again the classical damped harmonic x -oscillator and its time-reversed image, the y -oscillator, discussed above. The Hamiltonian of the $\{x, y\}$ -system can be schematically written as

$$H = H_I - H_{II} \quad \text{with the constraint } H_{II}|\psi\rangle = 0 , \quad (16)$$

in order to define physical states ψ and guarantee that the Hamiltonian is bounded from below. The physical consequence of this constraint is *information loss*. Physical states are invariant under time reversal and periodical, implying that

$${}_H\langle\psi(\tau)|\psi(0)\rangle_H = e^{i\alpha\pi} , \quad (17)$$

where $\tau = 2\pi/\Omega$ (with Ω expressed in terms of m, k, γ) stands for the period, and α is a real constant. Hence

$$\langle\psi_n(\tau)|H|\psi_n(\tau)\rangle = \hbar\Omega(n + \alpha/2) = \hbar\Omega n + E_0 , \quad (18)$$

with $E_0 = (\hbar/2)\Omega\alpha$ the zero point ($n = 0$) energy. Note that the index n above signals the n -dependence of the state and the corresponding energy. In conclusion, the zero point quantum contribution to the spectrum of physical states found above, results from information loss, imposed by the underlying dissipative dynamics [8].

The algebra doubling can also lead to neutrino oscillations. Linking the algebra doubling to the deformed Hopf algebra, one can build Bogogliubov operators as linear combinations of the co-product operators defined in terms of the deformation parameter obtained from the doubled algebra, and show the emergence of neutrino mixing [9]. In particular, one can write the mixing transformations connecting the flavour fields ψ_f to the neutrino fields with nonvanishing masses ψ_m as

$$\nu_e(x) = G_\theta^{-1}(t)\nu_1(x)G_\theta(t) \quad ; \quad \nu_\mu(x) = G_\theta^{-1}(t)\nu_2(x)G_\theta(t) , \quad (19)$$

through the generator of field mixing transformations $G_\theta(t)$. Note that for simplicity, and no loss of generality, we have only used two neutrino species. Then writing ψ_m in terms of flavour

creation/annihilation operators, and similarly writing ψ_m in terms of mass creation/annihilation operators, one finds that $G_\theta(t)$ contains rotation operator terms and Bogogliubov transformation operator terms. Since deformed co-products are a basis of Bogogliubov transformations, one concludes that field mixing arises from the algebraic structure of the deformed co-product in the noncommutative Hopf algebra embedded in the algebra doubling of noncommutative spectral geometry. We can hence conclude that the SM derived from NCSG, includes neutrino mixing by construction [9].

4. NCSG leading to an extended gravitational theory

We are currently living in a very exciting time for early universe cosmology, since our models can be now tested with a variety of very precise astrophysical and high energy physics data, and in particular with the Cosmic Microwave Background temperature anisotropies data and the Large Hadron Collider results. However, despite the present golden era of cosmology, a number of questions are still awaiting for a definite answer. For instance, one does not know the origin of dark matter and dark energy, whilst the search for a natural and well-motivated inflationary model (or plausible alternatives to the inflationary paradigm) still remains unsuccessful.

The main approaches to build early universe cosmological models have been based to string/M-theory or some non perturbative approach to Quantum Gravity, with Loop Quantum Cosmology being the leading candidate. Noncommutative spectral geometry can provide another proposal, since the model lives by construction at the GUT scale.

The bosonic action in Euclidean signature, favoured by the formalism of spectral triples, is [3]

$$\begin{aligned} \mathcal{S}^E = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \right. \\ \left. + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \frac{1}{2} |D_\mu \mathbf{H}|^2 - \mu_0^2 |\mathbf{H}|^2 - \xi_0 R |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \right) \sqrt{g} d^4x , \end{aligned} \quad (20)$$

where

$$\begin{aligned} \kappa_0^2 &= \frac{12\pi^2}{96f_2\Lambda^2 - f_0\mathfrak{c}} \quad , \quad \alpha_0 = -\frac{3f_0}{10\pi^2} \quad , \\ \gamma_0 &= \frac{1}{\pi^2} \left(48f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d} \right) \quad , \quad \tau_0 = \frac{11f_0}{60\pi^2} \quad , \\ \mu_0^2 &= 2\Lambda^2 \frac{f_2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \quad , \quad \xi_0 = \frac{1}{12} \quad , \\ \lambda_0 &= \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2} \quad , \quad \mathbf{H} = (\sqrt{af_0}/\pi)\phi \quad ; \end{aligned} \quad (21)$$

\mathbf{H} a rescaling of the Higgs field ϕ to normalize the kinetic energy, and the momentum f_0 is physically related to the coupling constants at unification. The geometric parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ correspond to the (running) Yukawa parameters of the particle physics model and the Majorana terms for the right-handed neutrinos. The first two terms in Eq. (20) depend only on the Riemann curvature tensor. The first is the Einstein-Hilbert term and the second is the Weyl curvature term; hence they are the Riemannian curvature terms. The third one is the cosmological term, while the fourth term

$$R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta} \quad ,$$

is the topological term that integrates to the Euler characteristic, hence is nondynamical. The three next terms are the Yang-Mills terms. The eighth term is the scalar minimal coupling term,

the next one is the scalar mass term, and the last one is the scalar quartic potential term. There is one more term, the $-\xi_0 R |\mathbf{H}|^2$, that couples gravity with the SM. For $\xi_0 = 1/12$, this term encodes the conformal coupling between the Higgs field and the Ricci curvature.

Hence, the Lagrangian obtained through the NCSG approach contains, in addition to the full SM Lagrangian, the Einstein-Hilbert action with a cosmological term, a topological term related to the Euler characteristic of the space-time manifold, a conformal Weyl term and a conformal coupling of the Higgs field to gravity. Within the NCSG context the Higgs field appears as the vector boson of the internal noncommutative degrees of freedom.

At this point, let us make a few remarks. The relations given in Eq. (21) above, are tied to the cutoff scale Λ , hence *a priori* there is no reason for those to hold at any other scale. Since the action Eq. (20) includes only the first three terms in the asymptotic expansion, one must be cautious keeping in mind that there are scales for which the neglected nonperturbative effects become important. Since to study physical consequences of the NCSG proposal one must use a Lorentzian signature, we will assume that a Wick rotation to imaginary time can be achieved. Noticing the absence of quadratic terms in the curvature — there is only the term quadratic in the Weyl curvature and the topological term $R^* R^*$ — we immediately conclude that for Friedmann-Lemaître-Robertson-Walker geometries, the Weyl term vanishes. Finally, notice the term that couples gravity with the SM, a term which should always be present when one considers gravity coupled to scalar fields.

The gravitational part of the asymptotic formula for the bosonic sector of the NCSG, including the coupling between the Higgs field and the Ricci curvature scalar, in Lorentzian signature, reads

$$\mathcal{S}_{\text{grav}}^L = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^* R^* - \xi_0 R |\mathbf{H}|^2 \right) \sqrt{-g} d^4x . \quad (22)$$

It will lead to the following equations of motion [10]:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{1}{\beta^2} \delta_{\text{cc}} \left[2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = \kappa_0^2 \delta_{\text{cc}} T_{\text{matter}}^{\mu\nu} , \quad (23)$$

where

$$\beta^2 \equiv -\frac{1}{4\kappa_0^2 \alpha_0} \quad \text{and} \quad \delta_{\text{cc}} \equiv [1 - 2\kappa_0^2 \xi_0 \mathbf{H}^2]^{-1} . \quad (24)$$

The definition δ_{cc} captures the conformal coupling between the Ricci scalar and the Higgs field.

In the low energy weak curvature regime, one may neglect the nonminimal coupling term between the background geometry and the Higgs field, getting $\delta_{\text{cc}} = 1$. Hence, for a cosmological context, namely Friedmann-Lemaître-Robertson-Walker space-time, the Weyl tensor vanishes and the noncommutative spectral geometry corrections to the Einstein equation vanish [10]. Consequently, any modifications to the background equation may appear at leading order only for anisotropic and inhomogeneous models, such as a Bianchi type-V model defined by the space-time metric

$$g_{\mu\nu} = \text{diag} \left[-1, \{a_1(t)\}^2 e^{-2nz}, \{a_2(t)\}^2 e^{-2nz}, \{a_3(t)\}^2 \right] , \quad (25)$$

where $a_i(t)$ with $i = 1, 2, 3$, arbitrary functions, denoting the three scale factors, and n an

integer. In this metric, the modified Friedmann equation reads [10]:

$$\begin{aligned}
& \kappa_0^2 T_{00} = \\
& -\dot{A}_3 \left(\dot{A}_1 + \dot{A}_2 \right) - n^2 e^{-2A_3} \left(\dot{A}_1 \dot{A}_2 - 3 \right) \\
& + \frac{8\alpha_0 \kappa_0^2 n^2}{3} e^{-2A_3} \left[5 \left(\dot{A}_1 \right)^2 + 5 \left(\dot{A}_2 \right)^2 - \left(\dot{A}_3 \right)^2 \right. \\
& \left. - \dot{A}_1 \dot{A}_2 - \dot{A}_2 \dot{A}_3 - \dot{A}_3 \dot{A}_1 - \ddot{A}_1 - \ddot{A}_2 - \ddot{A}_3 + 3 \right] - \frac{4\alpha_0 \kappa_0^2}{3} \sum_i \left\{ \dot{A}_1 \dot{A}_2 \dot{A}_3 \dot{A}_i \right. \\
& + \dot{A}_i \dot{A}_{i+1} \left(\left(\dot{A}_i - \dot{A}_{i+1} \right)^2 - \dot{A}_i \dot{A}_{i+1} \right) + \left(\ddot{A}_i + \left(\dot{A}_i \right)^2 \right) \left[-\ddot{A}_i - \left(\dot{A}_i \right)^2 + \frac{1}{2} \left(\ddot{A}_{i+1} + \ddot{A}_{i+2} \right) \right. \\
& \left. + \frac{1}{2} \left(\left(\dot{A}_{i+1} \right)^2 + \left(\dot{A}_{i+2} \right)^2 \right) \right] + \left[\ddot{A}_i + 3\dot{A}_i \ddot{A}_i - \left(\ddot{A}_i + \left(\dot{A}_i \right)^2 \right) \left(\dot{A}_i - \dot{A}_{i+1} - \dot{A}_{i+2} \right) \right] \\
& \left. \times \left[2\dot{A}_i - \dot{A}_{i+1} - \dot{A}_{i+2} \right] \right\}, \quad (26)
\end{aligned}$$

where we have defined $A_i(t) = \ln a_i(t)$ with $i = 1, 2, 3$. One then immediately observes that the correction terms in Eq. (26) above come in two types. Those which are fourth order in time derivatives, and those that are at the same order as the ones derived from the standard Einstein-Hilbert action. The former can be considered as small corrections since usually in cosmology we have slowly varying functions. The latter are proportional to n^2 , hence they vanish for homogeneous versions of Bianchi type-V. In conclusion, the corrections to Einstein's equations are only present in inhomogeneous and anisotropic space-times [10].

As energies are approaching the Higgs scale, the nonminimal coupling of the Higgs field to the curvature can no longer be neglected. Then the equations of motion read [10]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa_0^2 \left[\frac{1}{1 - \kappa_0^2 |\mathbf{H}|^2 / 6} \right] T_{\text{matter}}^{\mu\nu}, \quad (27)$$

where for simplicity we set $\beta = 0$. We thus conclude that $|\mathbf{H}|$ leads to an effective gravitational constant. A different way to see the rôle of the nonminimal coupling is by examining its effect on the equations of motion for the Higgs field in a constant gravitational field. Hence, since

$$-\mu_0 |\mathbf{H}|^2 \rightarrow -\left(\mu_0 + \frac{R}{12} \right) |\mathbf{H}|^2, \quad (28)$$

we conclude that for constant curvature, the self interaction of the Higgs field is increased.

Finally, one may find links to dilatonic gravity and chameleon cosmology [10]. Firstly, the action

$$\mathcal{L}_{|\mathbf{H}|} = -\frac{R}{12} |\mathbf{H}|^2 + \frac{1}{2} |D^\alpha \mathbf{H}| |D^\beta \mathbf{H}| g_{\alpha\beta} - \mu_0 |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \quad (29)$$

(where D^α denotes covariant derivative) for the pure Higgs field \mathbf{H} , can be written in the form of four-dimensional dilatonic gravity as

$$\mathcal{L}_{\tilde{\phi}} = e^{-2\tilde{\phi}} \left[-R + 6D^\alpha \tilde{\phi} D^\beta \tilde{\phi} g_{\alpha\beta} - 12 \left(\mu_0 - 12\lambda_0 e^{-2\tilde{\phi}} \right) \right], \quad (30)$$

by a redefinition of the Higgs field as

$$\tilde{\phi} = -\ln \left(|\mathbf{H}| / (2\sqrt{3}) \right). \quad (31)$$

Secondly, chameleon models are characterised by the existence of a scalar field having a nonminimal coupling to the standard matter content (thus evading solar system tests of GR). In the context of NCSG, the Higgs field has a nonzero coupling to the background geometry. If the equations of motion can be approximated by Einstein's equations, then the background geometry will be approximately given by standard matter, making the mass and dynamics of the Higgs field explicitly dependent of the local matter content.

Exploring the possible rôle of scalar fields appearing in the NCSG action, one may wonder whether the Higgs field, through its nonminimal coupling to the background geometry, can be the inflaton. The Gravity-Higgs sector of the asymptotic expansion of the spectral action, in Lorentzian signature reads

$$S_{\text{GH}}^{\text{L}} = \int \left[\frac{1 - 2\kappa_0^2 \xi_0 H^2}{2\kappa_0^2} R - \frac{1}{2} (\nabla H)^2 - V(H) \right] \sqrt{-g} \, d^4x , \quad (32)$$

where

$$V(H) = \lambda_0 H^4 - \mu_0^2 H^2 , \quad (33)$$

with μ_0 and λ_0 subject to radiative corrections as functions of energy. For each value of the top quark mass, there is a value of the Higgs mass where the Higgs potential is locally flattened [13]. However, since the flat region is narrow, the slow-roll must be very slow, otherwise the quasi-exponential expansion will not last long enough. Moreover, the amplitude of density perturbations in the Cosmic Microwave Background must be in agreement with the measured one.

Calculating the renormalisation of the Higgs self-coupling up to two-loops and constructing an effective potential which fits the renormalisation group improved potential around the flat region, one concludes that while the Higgs potential can lead to the slow-roll conditions being satisfied, the constraints imposed from the CMB data make the predictions of such a scenario incompatible with the measured value of the top quark mass [13].

The gravitational sector of the NCSG action provides a proposal for an extended theory of gravity. Studying the astrophysical consequences of such a theory, one is able to constrain one of the three momenta of the cutoff function, namely f_0 (or equivalently $a_0 = -3f_0/(10\pi^2)$ in Eq. (22)) which specifies the initial conditions on the gauge couplings [14]-[18]. Hence we will get a restriction on the particle physics at unification. Note that one cannot constrain the other two free parameters (the momenta f_2, f_4) without an *ad hoc* assumption on the running of the coefficients in the action functional.

To simplify the analysis and with no loss of generality, let us neglect in the following the conformal coupling between the Ricci curvature and the Higgs field. Hence to find nonzero correction terms we have to go beyond the homogeneous and isotropic case. The equations of motion read [10]

$$G^{\mu\nu} + \frac{1}{\beta^2} [2\nabla_\lambda \nabla_\kappa C^{\mu\nu\lambda\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = \kappa^2 T_{(\text{matter})}^{\mu\nu} , \quad (34)$$

where $\kappa^2 \equiv 8\pi G$, $G^{\mu\nu}$ is the (zero order) Einstein tensor, $T_{\text{matter}}^{\mu\nu}$ the energy-momentum tensor of matter and $\beta^2 = 5\pi^2/(6\kappa^2 f_0)$. Performing a detailed analysis of linear perturbations

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} , \quad (35)$$

around a Minkowski background metric $\eta_{\mu\nu}$, one can show that the linearised equation of motion, derived within the NCSG context, reads [10]

$$\left(1 - \frac{1}{\beta^2} \square_\eta \right) \square_\eta \bar{h}^{\mu\nu} = -2\kappa^2 T_{\text{matter}}^{\mu\nu} , \quad (36)$$

with $T_{\text{matter}}^{\mu\nu}$ taken to lowest order in $\gamma^{\mu\nu}$, so that it is independent of $\gamma^{\mu\nu}$ and satisfies the conservation equation $\partial_\mu T_{(\text{matter})}^{\mu\nu} = 0$. The equation of motion above has been written in terms of the tensor [15]

$$\bar{h}_{\mu\nu} = \bar{\gamma}_{\mu\nu} - \frac{1}{3\beta^2} \mathcal{Q}^{-1} (\eta_{\mu\nu} \square_\eta - \partial_\mu \partial_\nu) \gamma, \quad (37)$$

with

$$\mathcal{Q} \equiv 1 - \frac{1}{\beta^2} \square_\eta, \quad (38)$$

and having defined $\bar{\gamma}_{\mu\nu}$ the *trace reverse* of $\gamma_{\mu\nu}$:

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma. \quad (39)$$

Note that we constrain $\alpha_0 < 0$, hence $\beta^2 > 0$, so that Minkowski is a stable vacuum of the theory. The general solution [15]

$$h^{\mu\nu} = 2\beta^2 \kappa \int dS(x') G_R(x, x') T^{\mu\nu}(x'), \quad (40)$$

to Eq. (36) is given in terms of Green's functions $G_R(x, x')$ which satisfy the fourth-order partial differential equation

$$(\square - \beta^2) \square G_R(x, x') = 4\pi \delta^{(4)}(x - x'); \quad (41)$$

the operators \square above are acting on x . Consequently, the field is given by [15]

$$h^{\mu\nu}(\mathbf{r}, t) = \frac{4G\beta}{c^4} \int d\mathbf{r}' dt' \frac{\Theta(T)}{\sqrt{(cT)^2 - |\mathbf{R}|^2}} \mathcal{J}_1 \left(\beta \sqrt{(cT)^2 - |\mathbf{R}|^2} \right) T^{\mu\nu}(\mathbf{r}', t') \Theta(cT - |\mathbf{R}|), \quad (42)$$

where $J_1(x)$ is the first order Bessel function of the first kind; Θ is the Heavyside step function; $T = t - t'$ is the difference between the time of observation t and time of emission t' of the perturbation; and $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ denotes the difference between the location \mathbf{r} of the observer and the location \mathbf{r}' of the emitter. One can then calculate the propagation of gravitational waves and investigate discrepancies from the results obtained within standard GR.

In the far-field limit $|\mathbf{r}| \approx |\mathbf{r} - \mathbf{r}'|$, the spatial components of the $h^{\mu\nu}$ field are [15]

$$h^{ik}(\mathbf{r}, t) \approx \frac{2G\beta}{3c^4} \int_{-\infty}^{t - \frac{1}{c}|\mathbf{r}|} \frac{dt'}{\sqrt{c^2(t - t')^2 - |\mathbf{r}|^2}} \mathcal{J}_1 \left(\beta \sqrt{c^2(t - t')^2 - |\mathbf{r}|^2} \right) \ddot{D}^{ik}(t'), \quad (43)$$

where D^{ik} stands for the quadrupole moment,

$$D^{ik}(t) \equiv \frac{3}{c^2} \int d\mathbf{r} x^i x^k T^{00}(\mathbf{r}, t). \quad (44)$$

In this limit, the rate of energy loss from a circular binary system, under the assumption that the internal structure of the pair of masses m_1 and m_2 can be neglected, reads

$$-\frac{d\mathcal{E}}{dt} \approx \frac{c^2}{20G} |\mathbf{r}|^2 \dot{h}_{ij} \dot{h}^{ij}, \quad (45)$$

where the time derivatives of the spatial components of the field are

$$\dot{h}^{ij} = \frac{4G\beta A^{ij}\omega^{ij}}{3c^4} \left[\sin(\omega^{ij}t + \phi^{ij}) f_c\left(\beta|\mathbf{r}|, \frac{\omega^{ij}}{\beta c}\right) + \cos(\omega^{ij}t + \phi^{ij}) f_s\left(\beta|\mathbf{r}|, \frac{\omega^{ij}}{\beta c}\right) \right], \quad (46)$$

(note that no summation is implied) and we have defined the functions $f_s(x, z)$, $f_c(x, z)$ as

$$\begin{aligned} f_s(x, z) &\equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \sin\left(z\sqrt{s^2 + x^2}\right), \\ f_c(x, z) &\equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \cos\left(z\sqrt{s^2 + x^2}\right). \end{aligned} \quad (47)$$

The integrals in Eq. (47) show a strong resonance behaviour at $z = 1$ corresponding to a critical frequency

$$2\omega_c = c\beta = c(-\alpha_0 G)^{-1}, \quad (48)$$

close to which strong deviations from the standard results of GR are expected; these integrals are easily evaluated numerically for $z > 1$ and $z < 1$.

In the large $|\mathbf{r}|$ limit, the rate of energy loss to gravitational radiation by a circular (for simplicity) binary system of masses m_1, m_2 (we denote by μ the reduced mass of the system) at a separation vector of magnitude ρ reads [14]

$$-\frac{d\mathcal{E}}{dt} \approx \frac{32G\mu^2\rho^4\omega^6}{5c^5} \times \begin{cases} 1 + \frac{C}{\beta|\mathbf{r}|(1-\frac{\omega}{\omega_c})} \mathcal{J}_1\left(\beta|\mathbf{r}| - \frac{\omega}{\omega_c}\right) + \dots & ; \omega < \omega_c \\ 4\sin^2\left(\beta|\mathbf{r}|\tilde{f}\left(\frac{\omega}{\omega_c}\right)\right) & ; \omega > \omega_c \end{cases}, \quad (49)$$

where in the $\omega < \omega_c$ case the dots refer to higher powers of $1/(\beta|\mathbf{r}|)$.

Hence, for orbital frequencies small compared to the critical one ω_c , any deviation from the standard GR result is suppressed by the distance to the source. In this case, the $\beta \rightarrow \infty$ limit leads, as expected, to the GR result. This is not the case however for $\omega > \omega_c$, since then the GR result is only recovered if $\beta|\mathbf{r}|\tilde{f}(\omega/\omega_c) = \pi/3$. We will thus only consider the $\omega < \omega_c$ physically interesting case, and restrict β (equivalently f_0) by requiring that the energy lost to gravitational radiation agrees with the one predicted by GR to within observational uncertainties. Note that the presence of the Bessel function implies that the amplitude of the deviation from the result obtained within standard GR will oscillate with frequencies as well as distances, however the effect will be suppressed by the $|\mathbf{r}|^{-1}$ factor.

Considering binary pulsar systems, for which the rate of change of the orbital frequency is well known, the observational constraint is $\beta \gtrsim 7.55 \times 10^{-13} \text{m}^{-1}$ [14]. This (weak) limit can be improved through future observations of rapidly orbiting binaries relatively close to the Earth.

One can also set a lower bound on the Weyl term appearing in the NCSG action using results from the Gravity Probe B [17] and LAsER RELativity Satellite (LARES) [18] experiments. Gravity Probe B satellite contains a set of four gyroscopes and has tested the geodesic and the frame-dragging (Lense-Thirring) effects of GR with extreme precision. The LARES mission is designed to test these two effects to within 1% of the value predicted within the theory of GR.

Let us write the metric in terms of the metric potentials Φ, Ψ and the vector potential \mathbf{A} as

$$ds^2 = -(1 + 2\Phi)dt^2 + 2\mathbf{A} \cdot d\mathbf{x}dt + (1 + 2\Psi)d\mathbf{x}^2. \quad (50)$$

The rate of an orbiting gyroscope precession can be then splitted into a part generated by the metric potentials and one generated by the vector potential. The obtained spin equation of motion for the gyro-spin three-vector is hence expressed as the sum of the instantaneous geodesic

and Lense-Thirring precessions. Each of these two precessions can be then written as the sum of two terms, one obtained within GR and one coming from NCSG. Setting the geodesic precession (equivalently for the Lense-Thirring precession) to be the one predicted from standard GR and requiring that the NCSG contribution is within the accuracy of its measured value, Gravity Probe B results imply [17] $\beta \gtrsim 7.1 \times 10^{-5} \text{m}^{-1}$, and LARES experiment sets [18] $\beta \gtrsim 1.2 \times 10^{-6} \text{m}^{-1}$.

A much stronger constraint can be imposed to β using the torsion balance experiments. The modifications induced by the NCSG action to the Newtonian potentials Φ, Φ lead to the following expressions for the components of $\gamma_{\mu\nu}$ [17]:

$$\begin{aligned}\gamma_{00} &= -2\Phi = \frac{2GM}{r} \left(1 - \frac{4}{3} e^{-\beta r} \right), \\ \gamma_{0i} &= \gamma_{i0} = A_i = -\frac{4G}{r^3} [1 - (1 + \beta r) e^{-\beta r}] (\mathbf{r} \wedge \mathbf{J})_i, \\ \gamma_{ij} &= 2\Psi \delta_{ij} = \frac{2GM}{r} \left[1 + \frac{5}{9} e^{-\beta r} \right] \delta_{ij}.\end{aligned}\tag{51}$$

These modifications are similar to those induced by a fifth-force through a potential

$$V(r) = -\frac{GMm}{r} \left(1 + \alpha e^{-r/\lambda} \right),\tag{52}$$

where α is a dimensionless strength parameter and λ a length scale. The tightest constraint on λ from the latest torsion balance experiments is $\lambda \lesssim 10^{-4} \text{m}$ leading to [17]

$$\beta \gtrsim 10^4 \text{m}^{-1},\tag{53}$$

a much stronger constraint than the ones obtained through pulsar timings, Gravity Probe B or LARES experiments.

5. A new approach: The zeta function regularisation

The cutoff bosonic spectral action is a quite successful and promising scheme, worth to be further investigated. Based upon an elegant mathematical theory, it offers a description of geometry in terms of spectral properties of operators and leads to a model of particle interactions which is very close to the real phenomenology as revealed from high energy physics experiments. Whilst the Standard Model and the Pati-Salam gauge groups fit into the NCSG model, the SU(5) or SO(10) groups do not; and absence of large groups is interesting since it prevents proton decay. Following the NCSG scheme, one is able to infer quantities related to the Higgs boson based only upon the input from the fermionic parameters in the fluctuated Dirac operator \mathcal{D}_A , which defines also the fermionic part of the bosonic spectral action.

However, despite its success, the cutoff spectral action faces some issues. It is calculated via the asymptotic heat kernel expansion and is only valid when the fields and their derivatives are small with respect to the cutoff energy scale Λ . Hence, the asymptotic expansion leading to the appearance of only three of the momenta of the cutoff function f is only valid in the weak-field approximation and one may wonder what does it happen in the ultraviolet regime when high momenta are the dominant ones. Issues with super-renormalisability have been addressed in the literature, indicating that high energy bosons do not propagate [26]. In short, within the traditional cutoff bosonic spectral action approach, it is not clear what is the meaning of the cutoff scale Λ nor what does it happen at scales beyond Λ . Moreover, the cutoff bosonic spectral action depends (even though not in a very strong matter) on the particular choice of the cutoff f function. Finally, there is an issue with the magnitude of the dimensionful parameters in the model, namely the cosmological constant, the Higgs vacuum expectation value and the

gravitational coupling. The natural value for the cosmological constant obtained through the spectral action approach is $\sim \Lambda^4$, which is clearly much bigger than its observational value. Thus, to render it compatible with the observational value of the cosmological constant, one should add by hand an appropriate term. The heat expansion does not lead to a minimum of the Higgs potential for all natural choices of the cutoff function. Hence one must add by hand to the H^2 term, a quadratic term with a large coefficient, to provide a minimum of the potential with a Higgs vacuum expectation value which is many orders of magnitude smaller than Λ . There is also a problem with the value of the gravitational constant given by the coefficient in front of the scalar curvature. The value obtained through the cutoff bosonic spectral action is at least one order of magnitude smaller than its experimental value. Therefore, one has again to add by hand an appropriate term. In conclusion, the physical values of these dimensionful parameters necessitate an experimental input beyond the NCSG approach. A different way of phrasing this problem is by calling it *the naturalness problem*.

To cure the dependence on the cutoff scale and the cutoff function, it has been recently proposed [19] another way to regularise the infinite sum of the eigenvalues of the (unbounded) fluctuated Dirac operator, based on the ζ function. More precisely, the zeta bosonic spectral action can be defined as [19]

$$S_\zeta \equiv \lim_{s \rightarrow 0} \text{Tr } \mathcal{D}^{-2s} \equiv \zeta(0, \mathcal{D}^2) , \quad (54)$$

with the zeta function given by the a_4 heat kernel coefficient associated with the Laplace type operator \mathcal{D}^2 :

$$S_\zeta = a_4 [\mathcal{D}^2] = \int d^4x \sqrt{g} L \quad \text{with} \quad L(x) = a_4(\mathcal{D}^2, x) . \quad (55)$$

This ζ spectral function leads to the following Lagrangian density [19]:

$$\begin{aligned} L(x) = & \alpha_1 M^4 + \alpha_2 M^2 R + \alpha_3 M^2 H^2 + \alpha_4 B_{\mu\nu} B^{\mu\nu} + \alpha_5 W_{\mu\nu}^\alpha W^{\mu\nu\alpha} + \alpha_6 G_{\mu\nu}^a G^{\mu\nu a} \\ & + \alpha_7 H \left(-\nabla^2 - \frac{R}{6} \right) H + \alpha_8 H^4 + \alpha_9 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \alpha_{10} R^* R^* . \end{aligned} \quad (56)$$

Note that $B_{\mu\nu}$, $W_{\mu\nu}$ and $G_{\mu\nu}$ stand for the field strength tensors of the corresponding U(1), SU(2) and SU(3) gauge fields; $\alpha_1, \dots, \alpha_{10}$ are dimensionless constants determined by the Dirac operator; $R^* R^*$ is the Gauss-Bonnet density and C is the Weyl tensor.

Up to this point, the dimensionful quantity M appearing in the position corresponding to the Majorana mass of the right-handed neutrino in the Dirac operator, is just a constant. All dimensionless constants are normalised to their spectral action values, whilst the three dimensionful parameters are normalised from experiments. In analogy with the cutoff bosonic spectral action we consider the zeta bosonic spectral action as valid at the scale $\Lambda \sim (10^{14} - 10^{17})$ GeV, but emphasising that the action is itself independent of the scale Λ . Since the zeta spectral action Eq. (56) does not contain higher than 4-dimensional operators, it is renormalisable and local. Moreover, there are no issues about asymptotic expansion or convergence, and the zeta spectral action is purely spectral with no dependence on a cutoff function.

Within the cutoff spectral action approach, if the full momentum-dependence of the propagators is considered, then the spectral dimension becomes independent of the spin and vanishes identically [27]. A physical interpretation of this behaviour can be summarised in the statement that *high energy bosons do not propagate* [26]. However, the zeta spectral action leads to nontrivial spectral dimensions. The Higgs scalar part of the bosonic action has the same behaviour in the ultraviolet as in the infrared limit, so that the spectral dimension coincides with the topological dimension of the manifold. The same holds for the gauge fields. Hence the

spectral dimension of the Higgs scalar and the gauge fields is equal to four. To calculate the gravitational spectral dimension one needs to do an analytic continuation [19]; we will highlight the computation below.

The spectral dimension D_s is defined as

$$D_s \equiv \lim_{T \rightarrow 0} \left[-2 \frac{\partial \log P(T)}{\partial \log T} \right], \quad (57)$$

where $P(T)$ stands for the value of the heat kernel $P(T, x, x')$, corresponding to the quadratic part of the gravitational part of the action for transverse and traceless fluctuations $h_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$, at $x = x'$, and T denotes the diffusion time. Such a heat kernel is given by the integral

$$P(T, x, x') = \int \frac{d^4 p}{(2\pi^4)} e^{ip(x-x')} e^{-(p^2 - ap^4)T}, \quad (58)$$

and one can show that there exists an analytic continuation of the relevant integral in a region of positive a [19]. The obtained gravitational spectral dimension for all nonzero real a is [19]

$$D_s = 2; \quad (59)$$

in agreement with the fact that here the gravitational propagators decrease faster in infinity due to the presence of fourth derivatives. Finally, there exists a *low energy* limit of the gravitational spectral dimension, valid for all real a , with [19]

$$D_s^{\text{low}} = 4, \quad (60)$$

as expected since at very low energies the dynamics does not feel the fourth derivative terms.

6. Conclusions

Noncommutative spectral geometry offers a geometric framework for the description of the Standard Model of strong and electroweak interactions, based upon a purely algebraic description. The constructed spectral action for fermions and bosons based on the spectral properties of the generalised Dirac operator and for an appropriately chosen algebra led to phenomenological results very close to the experimental ones.

The doubling of the algebra, which can be interpreted as considering a geometric space formed by two copies of a four-dimensional manifold, has profound physical implications. In particular, the doubling of the algebra is required in order to incorporate gauge symmetries, a fundamental ingredient of the Standard Model, and it is also the main element to explain neutrino mixing. Following 't Hooft's conjecture, one can also show that the NCSG classical construction carries implicit in its doubling of the algebra the seeds of quantisation.

Considering the gravitational sector of the spectral action, one can constrain one of the momenta of the cutoff function, namely the one related to the coupling constants at unification. The strongest constraint on the coefficient of the curvature squared term is obtained through torsion balance experiments, while using data from binary pulsars, GPB and LARES experiments the corresponding constraint is much weaker.

To address the issues of renormalisability and spectral dimensions, a novel definition of the bosonic spectral action has been proposed, based upon the ζ function regularisation. While the zeta spectral action shares the same predictive power with the traditional cutoff spectral action, only the former leads to a local, unitary and renormalisable theory. In this new proposal, the open aspect that remains to be addressed is the dynamical generation of the three dimensionful fundamental constants, namely the cosmological constant, the Higgs vacuum expectation value, and the gravitational constant.

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